Modeling Bond Prices In Continuous-Time
Part I - Risk-Free Bond Price, Duration and Convexity

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November, 2020

In this white paper we will build a model that calculates the market price of a risk-free bond in continous-time and then estimates the change in bond price with respect to a change in discount rate (i.e. Duration and Convexity).

Our Hypothetical Problem

The table below presents our go-forward model assumptions...

Table 1: Bond Assumptions

Symbol	Description	Balance
B	Bond face value	\$1,000
C	Annual coupon rate (%)	4.50
T	Term in years (#)	3.00

We are tasked with answering the following questions:

Question 1: What is bond price given a continuous-time discount rate of 6.00% and 7.00%, respectively?

Question 2: Use duration and convexity to estimate the change in bond price in the question above.

Bond Price Equations

We will define the variable P_0 to be the price at time zero of a coupon paying bond and the variable κ to be the continuous-time discount rate. Using Table 1 above the equation for bond price at time zero is...

$$P_0 = \int\limits_0^T C\,B\,\mathrm{Exp}\left\{-\kappa\,u\right\}\delta u + B\,\mathrm{Exp}\left\{-\kappa\,T\right\} \tag{1}$$

Note that we can rewrite Equation (1) above is...

$$P_0 = B\left(C\int\limits_0^T \mathrm{Exp}\left\{-\kappa\,u\right\}\delta u + \mathrm{Exp}\left\{-\kappa\,T\right\}\right) \tag{2}$$

Using Appendix Equation (17) below the solution to Equation (2) above is...

$$P_0 = B\left(C\,\kappa^{-1} - C\,\mathrm{Exp}\left\{-\kappa\,T\right\}\kappa^{-1} + \mathrm{Exp}\left\{-\kappa\,T\right\}\right) \tag{3}$$

Using Equation (3) above the first derivative of bond price with respect to discount rate is...

$$\frac{\delta}{\delta\kappa}P_0 = B\left(C\,\frac{\delta}{\delta\kappa}\,\kappa^{-1} - C\,\frac{\delta}{\delta\kappa}\,\mathrm{Exp}\left\{-\kappa\,T\right\}\kappa^{-1} + \frac{\delta}{\delta\kappa}\,\mathrm{Exp}\left\{-\kappa\,T\right\}\right) \tag{4}$$

Using Equation (3) above the second derivative of bond price with respect to discount rate is...

$$\frac{\delta^2}{\delta\kappa^2}P_0 = B\left(C\,\frac{\delta^2}{\delta\kappa^2}\,\kappa^{-1} - C\,\frac{\delta^2}{\delta\kappa^2}\,\mathrm{Exp}\left\{-\kappa\,T\right\}\kappa^{-1} + \frac{\delta^2}{\delta\kappa^2}\,\mathrm{Exp}\left\{-\kappa\,T\right\}\right) \tag{5}$$

Change in Bond Price

Using Equations (4) and (5) above we will define duration and convexity to be the following equations...

$$\text{Duration} = \frac{\delta}{\delta\kappa}\, P_0 \Big/ B \;\; \text{....and...} \;\; \text{Convexity} = \frac{\delta^2}{\delta\kappa^2}\, P_0 \Big/ B \tag{6}$$

Using Equation (6) above the equation for the change in bond price with respect to a change in discount rate via a Taylor Series Expansion of order two is...

$$\delta\, P_0 = \Bigg(B \times \text{Duration} \times \delta\kappa\Bigg) + \Bigg(B \times \frac{1}{2} \times \text{Convexity} \times \delta\kappa^2\Bigg) \tag{7}$$

Using Equation (4) above and Appendix Equations (18), (19) and (20) below the solution to the duration equation in Equations (6) and (7) above is...

$$\begin{aligned}
\text{Duration} &= \frac{\delta}{\delta\kappa}\, P_0 \Big/ B \\
&= \Bigg[-C\,\kappa^{-2} + C\,\text{Exp}\Bigg\{-\kappa\, T\Bigg\}\Bigg(1 + \kappa\, T\Bigg)\,\kappa^{-2} - T\,\text{Exp}\Bigg\{-\kappa\, T\Bigg\}\Bigg] \\
&= \Bigg[-C\Bigg(1 + \text{Exp}\Bigg\{-\kappa\, T\Bigg\}\Bigg(1 + \kappa\, T\Bigg)\Bigg)\kappa^{-2} - T\,\text{Exp}\Bigg\{-\kappa\, T\Bigg\}\Bigg] \\
&= -\Bigg[C\Bigg(1 + \text{Exp}\Bigg\{-\kappa\, T\Bigg\}\Bigg(1 + \kappa\, T\Bigg)\Bigg)\kappa^{-2} + T\,\text{Exp}\Bigg\{-\kappa\, T\Bigg\}\Bigg]
\end{aligned} \tag{8}$$

Using Equation (4) above and Appendix Equations (18), (19) and (21) below the solution to the convexity equation in Equations (6) and (7) above is...

$$\begin{aligned}
\text{Convexity} &= \frac{\delta^2}{\delta\kappa^2}\, P_0 \Big/ B \\
&= \Bigg[2\,C\,\kappa^{-3} - C\,\text{Exp}\Bigg\{-\kappa\, T\Bigg\}\Bigg(\kappa^2 T^2 + 2\,\kappa\, T + 2\Bigg)\kappa^{-3} + T^2\,\text{Exp}\Bigg\{-\kappa\, T\Bigg\}\Bigg] \\
&= \Bigg[C\Bigg(2 - \text{Exp}\Bigg\{-\kappa\, T\Bigg\}\Bigg(\kappa^2 T^2 + 2\,\kappa\, T + 2\Bigg)\kappa^{-3} + T^2\,\text{Exp}\Bigg\{-\kappa\, T\Bigg\}\Bigg]
\end{aligned} \tag{9}$$

The Answers To Our Hypothetical Problem

Question 1: What is bond price given a continuous-time discount rate of 6.00% and 7.00%, respectively?

Bond price at a discount rate of 6.00% is...

$$P_0 = 1{,}000 \times \Bigg(0.045 \times 0.06^{-1} - 0.045 \times \text{Exp}\Bigg\{-0.06 \times 3\Bigg\} \times 0.06^{-1} + \text{Exp}\Bigg\{-0.06 \times 3\Bigg\}\Bigg) = 958.82 \tag{10}$$

Bond price at a discount rate of 7.00% is...

$$P_0 = 1{,}000 \times \Bigg(0.045 \times 0.07^{-1} - 0.045 \times \text{Exp}\Bigg\{-0.07 \times 3\Bigg\} \times 0.07^{-1} + \text{Exp}\Bigg\{-0.06 \times 3\Bigg\}\Bigg) = 932.35 \tag{11}$$

Question 2: Use duration and convexity to estimate the change in bond price in the question above.

$$\text{Model Parameters:} \;\; \kappa = 0.0600 \;\; \text{...and...} \;\; \delta\kappa = 0.0700 - 0.0600 = 0.0100 \tag{12}$$

Using the model parameters in Equation (12) above and duration Equation (9) above the equation for duration for our problem is...

$$\begin{aligned}
\text{Duration} = -\Bigg[&0.045 \times \Bigg(1 + \text{Exp}\Bigg\{-0.06 \times 3\Bigg\}\Bigg(1 + 0.06 \times 3\Bigg)\Bigg) \times 0.06^{-2} \\
&+ 3 \times \text{Exp}\Bigg\{-0.06 \times 3\Bigg\}\Bigg] = -2.6856
\end{aligned} \tag{13}$$

Using the model parameters in Equation (12) above and convexity Equation (**??**) above the equation for convexity for our problem is...

$$\text{Convexity} = \Bigg[0.045 \times \Bigg(2 - \text{Exp}\Bigg\{-0.06 \times 3\Bigg\}\Bigg(0.06^2 \times 3^2 + 2 \times 0.06 \times 3 + 2\Bigg) \times 0.06^{-3} + 3^2 \times \text{Exp}\Bigg\{-0.06 \times 3\Bigg\}\Bigg] = 7.8715 \tag{14}$$

Using Equations (7), (13) and (14) above the answer to our problem is...

$$\text{Change in Bond Price} = 1,000 \times -2.6856 \times 0.01 + 1,000 \times \frac{1}{2} \times 7.8715 \times 0.01^2 = -26.46 \tag{15}$$

Note that using Equations (10) and (11) above the actual change in bond price was...

$$\text{Change in Bond Price} = 932.35 - 958.82 = -26.47 \tag{16}$$

The answers (actual change in price vs estimated change in price) are not the same because the actual Taylor Series Expansion goes beyond order two and therefore using duration (order one) and convexity (order two) is just an estimate of the change in bond price and not the actual change.

Appendix

A. The solution to the following integral is...

$$\int_0^T \text{Exp}\Bigg\{-\kappa\, u\Bigg\}\,\delta u = -\kappa^{-1}\,\text{Exp}\Bigg\{-\kappa\, u\Bigg\}\Bigg[_0^T = \kappa^{-1}\Bigg(1 - \text{Exp}\Bigg\{-\kappa\, T\Bigg\}\Bigg) \tag{17}$$

B. The first and second derivative of the following equation is...

$$\frac{\delta}{\delta\kappa}\,\kappa^{-1} = -\kappa^{-2} \quad \text{...and...} \quad \frac{\delta^2}{\delta\kappa^2}\,\kappa^{-1} = 2\,\kappa^{-3} \tag{18}$$

C. The first and second derivative of the following equation is...

$$\frac{\delta}{\delta\kappa}\,\text{Exp}\Bigg\{-\kappa\, u\Bigg\} = -u\,\text{Exp}\Bigg\{-\kappa\, u\Bigg\} \quad \text{...and...} \quad \frac{\delta^2}{\delta\kappa^2}\,\text{Exp}\Bigg\{-\kappa\, u\Bigg\} = u^2\,\text{Exp}\Bigg\{-\kappa\, u\Bigg\} \tag{19}$$

D. Using Appendix Equation (18) and (19) above the first derivative of the following equation via the product rule is...

$$\frac{\delta}{\delta\kappa}\,\kappa^{-1}\,\text{Exp}\Bigg\{-\kappa\, u\Bigg\} = -\kappa^{-2}\,\text{Exp}\Bigg\{-\kappa\, u\Bigg\} - u\,\kappa^{-1}\,\text{Exp}\Bigg\{-\kappa\, u\Bigg\} = -\kappa^{-2}\,\text{Exp}\Bigg\{-\kappa\, u\Bigg\}\Bigg(1 + \kappa\, u\Bigg) \tag{20}$$

The second derivative of Equation (20) above is...

$$\frac{\delta^2}{\delta\kappa^2}\,\kappa^{-1}\,\text{Exp}\Bigg\{-\kappa\, u\Bigg\} = \kappa^{-3}\,\text{Exp}\Bigg\{-\kappa\, u\Bigg\}\Bigg(\kappa^2 u^2 + 2\,\kappa\, u + 2\Bigg) \tag{21}$$